Introduction

Algebra I Standard:

Interpret functions that arise in applications in terms of the context. [Linear, exponential, and quadratic]
4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

Historical Background:

Curves and points drawn on a two-dimensional surface can be represented with equations with respect to a pair of coordinates (today, usually chosen as perpendicular x and y coordinates). Though the two-dimensional surface is often called the Cartesian plane, this combination of algebra and geometry is due to the independent work of Ren Descartes (1596–1650) and Pierre Fermat (1601–1665)–both seventeenth century Frenchmen. In 1637 Descartes was living in the Netherlands, teaching in university there and writing philosophy, meanwhile Fermat lived in Toulouse and worked as a lawyer. Both corresponded with each other in an elaborate letter-exchange network coordinated by Marin Mersenne in Paris. Aspiring mathematicians would send their most recent work to Mersenne in letters that he would copy and distribute to relevant, interested parties.

In fact, Fermat's treatment of analytic geometry, in a text entitled *Ad locos planos et solidos isagoge (Introduction to Plane and Solid Loci)* was only circulated through Mersenne and never appeared in print during Fermat's lifetime. Nevertheless, the text was well-received among Mersenne's correspondents, and Descartes most likely first saw it just before his text, *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences (Discourse on the Method for Rightly Directing Ones Reason and Searching for Truth in the Sciences)* went to print.

Fermat and Descartes corresponded about their analytic geometry via Mersenne in 1638. The following script liberally adapts and embellishes certain excerpts of these letters (the French excerpts can be found in Grégoire (1998)) that indicate the importance of priority, the subtle differences perceived between the different approaches, and the contrasting styles of the two mathematicians. This exchange specifically concerns the process of finding a tangent line and extreme points (maximum or minimum) of a curve. Descartes' method only worked for conic sections (circles, parabolas, hyperbolas, ellipses), while Fermat's method worked for general curves and resembled calculus (which would be invented about 30 years later and cause another priority dispute!).

Script

Narrator: Pierre de Fermat was born in Toulouse and has never strayed far from home, where he works as a lawyer. His true passion, however, is mathematics. He sits at a desk surrounded by papers. He is in his late thirties, round about the edges, losing some hair. He is dressed soberly in black. His desk is full of papers and he seems very busy.

Fermat: My dear Mersenne, would you be so kind as to compare Descartes' method of tangents with my own as introduced in my little essay on maximum and minimum from 1629? As I understand it, his version only applies to conic sections, and so is less general than mine. As you know, my method involves beginning with a secant line and then gradually reducing the distance between the two points of intersection until they coincide. Thus the secant becomes a tangent! I have not yet been able to obtain a copy of his *Discours* down here in Toulouse, but my understanding in his work involves circles. I would love to spend the morning writing about math, but alas! I haven't the time. Thank you for your assistance! Pierre de Fermat.

Narrator: Only a few days later (mail was surprisingly fast in the 17th century!), Descartes learns of Fermat's method of tangents. Though living in the Netherlands, Descartes remains in contact with French mathematics through letters with Marin Mersenne in Paris. Descartes is visibly upset and paces, while writing.

Descartes: Mersenne! I am very astonished that Fermat is entering this competition with such bad weapons! His rule is nothing other than false position, based on a manner of demonstration which reduces to the impossible, and is less admirable and less ingenious than all those that one uses in Mathematics. Instead there is my method, drawn from a knowledge of the nature of Equations which has never been explained except in the third book of my Geometry text, and it pursues the noblest manner of proof that can be, that which is called *a priori*.

Narrator: Mersenne dutifully passes the news on to Fermat, who responds in turn.

Fermat: My dear Mersenne, I have learned from your letter than my reply to Monsieur Descartes was not appealing. Perhaps my rules having been presented unadorned and without demonstration, they have not been understood or they have appeared too easy to Monsieur Descartes who has made such inroads and has taken such a painful path for finding tangents in his geometry. I would not wish to send you anything more for Monsieur Descartes, since he places such severe laws on an innocent trade.

Narrator: Despite Fermat's promised silence, Descartes continues to be upset and is made more so when he learned that Fermat's method was catching on among other French mathematicians!

Descartes: Mersenne, please send this letter to Etienne Pascal and Roberval who have persisted in defending the method of Fermat against mine. As you can see from my notes, I have applied Fermat's method for finding a tangent of a parabola to the case of an ellipse. It does not work! If one promises that this reasoning is good for the parabola, one must also promise that it is good for the ellipse and the hyperbola, and all the other curved lines that are in the world, hence one immediately and clearly sees that he does not conclude the truth.

Narrator: Fermat realizes he had been misread and wrote to Mersenne to clarify.

Fermat: My dear Mersenne, if you will forgive this hasty note. To correct Monsieur Descartes' understandable mistake please note that in the case of an ellipse one must subtly modify the method to account for the different kind of curves. This is easily done as one can see from the attached equations. All the best.

Descartes: Mersenne, I have received Monsieur Fermat's response. It appears that he has found his rule only by groping blindly, at least he does not clearly conceive the principles.

Fermat: My dear Mersenne, it appears I must be more explicit. Please see the accompanying explanation of my method of tangents to any curves. I also explain at great length how to find the maximum and minimum points and distinguish between the two. If you would be so kind as to circulate these writings to Descartes, I would be very appreciative!

Narrator: By mid-year, Descartes at last seems satisfied and writes to Fermat directly.

Descartes: Dear Fermat, I am writing to you directly. I have had no less joy in receiving the letter in which you do me the favor of promising me your friendship, than if it came to me from a Mistress whose good graces I had passionately desired. And seeing the latter technique which you use to find tangents to curved lines, I have nothing to reply to that except that it is very good. That's it! Your friend Descartes.

Narrator: But then Descartes writes another letter.

Descartes: Mersenne. Between us, to tell the truth, I believe that if Fermat had not seen what I had asked him to correct, he would not have known how to disentangle himself. I also believe that all this chicanery about the line EB, whether it should be named the greatest, that his friends in Paris have kept up for half a year, was invented by them only to give him time to find something better to respond to me. It would not be a big deal to provide the tangents of certain curves–curves that he invented solely in order to be able to find their tangents–and which, moreover, are of no use. In this way I see nothing at all to admire in all his writing, despite the epithets of marvelous, excellent and miraculous, which he has given to things which are either very simple or just bad.

Narrator: And so the two mathematicians parted on amicable terms. The End.

Problem set

Typically one studies tangent lines in calculus, but as Descartes and Fermat show (they both worked before calculus was "invented"), this is not always necessary. A tangent to a parabola is defined as a line that intersects a parabola in exactly one point. We can find the points of intersection of a parabola and a straight line through algebraic substitution and the quadratic equation. Since a parabola and a straight line intersect in only one point, the discriminant in a quadratic equation (that's the part under the radical) has to be zero. These facts let us find the slope of a line tangent to a parabola at a given point without calculus. Descartes used a similar method, but with circles instead of straight lines.

Example

Find the tangent line to the parabola $y = x^2 - 3$ at the point $(1, -2)$.

The equation for the line through $(1, -2)$ is $y+2 = m(x-1)$. Solving both equations for y and substituting gives $m(x-1) - 2 = x^2 - 3$. Rewrite as a quadratic equation equal to zero, $x^2 - mx + (m-1) = 0$. The determinant of this quadratic is $b^2 - 4ac = m^2 - (4)(1)(m-1)$. This can be simplified to $m^2 - 4m + 4 = 0$. Factoring this as a quadratic, gives the single solution $m = 2$. So the line is $y + 2 = 2(x-1)$ or, written in slope-intercept form, $y = 2x - 4$

Practice problems

1. Find the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.
2. Find the tangent line to the parabola $y = -2x^2 + 4$ at the point $(2, 4)$.
3. Find the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $(0, 1)$.

Bibliography

Goldstein, Catherine. "Routine controversies : Mathematical challenges in Mersenne's correspondence." *Revue d'histoire des sciences* tome 66, no. 2, 2013. pp. 249-273.

Grabiner, Judith V. "Descartes and Problem-Solving." *Mathematical Magazine* volume 68, no. 2, April 1995. pp. 83-97.

Grégoire, Michèle. "La correspondance entre Descartes et Fermat/The correspondence between Descartes and Fermat." *Revue d'histoire des sciences*, tome 51, no. 2-3, 1998. pp. 355-362.